Solutions to Problem 1.

a. M/M/12 with $\lambda = 5$ customers / minute and $\mu = \frac{1}{2}$ customers / minute

b. $\rho = \dfrac{\lambda}{s\mu} = \dfrac{5}{6} \approx 0.8333$

c. We want π_0:

$$\pi_0 = \left[\left(\sum_{j=0}^{12} \frac{(\frac{5}{6}(12))^j}{j!} \right) + \frac{12^{12}(\frac{5}{6})^{13}}{12!(1-\frac{5}{6})} \right]^{-1} \approx 0.000036$$

d. We want w_q:

$$\begin{aligned}
\pi_{12} &= \frac{10^{12}}{12!}\pi_0 \approx 0.0749 \\
\Rightarrow \ell_q &= \frac{\pi_{12}(\frac{5}{6})}{(1-\frac{5}{6})^2} \approx 2.2469 \text{ customers} \\
\Rightarrow w_q &= \frac{\ell_q}{\lambda} \approx 0.4494 \text{ minutes}
\end{aligned}$$

Solutions to Problem 2.

a. We approximate both the one- and two-employee systems as M/M/1 queues.

- **One-employee system.** $\lambda = 24$ customers per hour, $\mu = 30$ customers per hour $\Rightarrow \rho = 0.8$

$$\begin{aligned}
\pi_0 &= \left[\left(\sum_{j=0}^{1} \frac{(0.8)^j}{j!} \right) + \frac{(0.8)^2}{(1-0.8)} \right]^{-1} = 0.2 \\
\pi_1 &= (0.8)(0.2) = 0.16 \\
\ell_q &= \frac{0.16(0.8)}{(1-0.8)^2} = 3.2 \text{ customers} \\
\ell &= 3.2 + 0.8 = 4 \text{ customers} \\
\Rightarrow w_q &= \frac{3.2}{24} \approx 0.1333 \text{ hours} \\
\Rightarrow w &= \frac{4}{24} \approx 0.1667 \text{ hours}
\end{aligned}$$

- **Two-employee system.** $\lambda = 24$ customers per hour, $\mu = 48$ customers per hour $\Rightarrow \rho = 0.5$

$$\begin{aligned}
\pi_0 &= \left[\left(\sum_{j=0}^{1} \frac{(0.5)^j}{j!} \right) + \frac{(0.5)^2}{(1-0.5)} \right]^{-1} = 0.5 \\
\pi_1 &= (0.5)(0.5) = 0.25 \\
\ell_q &= \frac{0.25(0.5)}{(1-0.5)^2} = 0.5 \text{ customers} \\
\ell &= 0.5 + 0.5 = 1 \text{ customer} \\
\Rightarrow w_q &= \frac{0.5}{24} \approx 0.0208 \text{ hours} \\
\Rightarrow w &= \frac{1}{24} \approx 0.0417 \text{ hours}
\end{aligned}$$

b. From above:

- **One-employee system.** $\pi_0 = 0.2$
- **Two-employee system.** $\pi_0 = 0.5$

c. The fraction of time that the intercom is blocked is $\sum_{j=3}^{\infty} \pi_j = 1 - \pi_0 - \pi_1 - \pi_2$. We have π_0 and π_1 above, we still need π_2.

- **One-employee system.**

$$\pi_2 = (0.8)^2(0.2) = 0.128$$
$$\Rightarrow 1 - \pi_0 - \pi_1 - \pi_2 = 1 - 0.2 - 0.16 - 0.128 = 0.512$$

- **Two-employee system.**

$$\pi_2 = (0.5)^2(0.5) = 0.125$$
$$\Rightarrow 1 - \pi_0 - \pi_1 - \pi_2 = 1 - 0.5 - 0.25 - 0.125 = 0.125$$

Solutions to Problem 3.

a. Assuming that the interarrival times and service times are exponentially distributed and there is no reneging, we can model this system as a M/M/2 queue with $\lambda = 20$ calls per hour, and $\mu = 20$ calls per hour.

b. To keep up with the calls, we want the largest λ such that

$$\rho = \frac{\lambda}{2\mu} < 1 \quad \Rightarrow \quad \lambda < 2\mu = 40 \text{ calls per hour}$$

So, the largest such $\lambda = 40$.

c. We want the largest λ such that

$$w_q = \frac{\ell_q}{\lambda} = \frac{\pi_2 \rho}{\lambda(1-\rho)^2} \le \frac{4}{60} \approx 0.0667 \text{ hours}$$

Using a computer and trial-and-error, the largest such $\lambda \approx 30$.

d. We want the largest λ such that

$$\sum_{j=8}^{\infty} \pi_j = 1 - \sum_{j=0}^{7} \pi_j \le 0.15$$

Using a computer and trial-and-error, the largest such $\lambda \approx 31$.

e. We formulate the system with reneging as a birth-death process:

- **State space.** $\mathcal{M} = \{0, 1, 2, \dots\}$
  Each state represents the number of customers, including those being served and those on hold.
- **Arrival rates.**

$$\lambda_i = 20 \quad \text{for } i = 0, 1, 2, \dots$$

- **Service rates.** Assuming the time to renege is exponential:

$$\mu_i = \begin{cases} 20i & \text{for } i = 1, 2 \\ 40 + 12(i-2) & \text{for } i = 3, 4, \dots \end{cases}$$